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# LINEAR AND NONLINEAR, SECOND-ORDER PROBLEMS WITH STURM-LIOUVILLE-TYPE, MULTI-POINT BOUNDARY CONDITIONS

BRYAN P. RYNNE

ABSTRACT. We consider the nonlinear boundary value problem consisting of the equation

$$-u'' = f(u), \quad \text{on } (-1, 1),$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, together with general Sturm-Liouville type, multi-point boundary conditions at  $\pm 1$ . We obtain Rabinowitz-type global bifurcation results, and then use these to obtain ‘nodal’ solutions of the problem. We conclude with a nonresonance result for an inhomogeneous form of the problem.

These results rely on the spectral properties of the eigenvalue problem consisting of the equation

$$-u'' = \lambda u, \quad \text{on } (-1, 1),$$

together with the multi-point boundary conditions. In a previous paper it was shown that, under certain ‘optimal’ conditions, the basic spectral properties of this eigenvalue problem are similar to those of the standard Sturm-Liouville problem with single-point boundary conditions. In particular, for each integer  $k \geq 0$  there exists a unique, simple eigenvalue  $\lambda_k$ , whose eigenfunctions have ‘oscillation count’ equal to  $k$ , where the ‘oscillation count’ was defined in terms of a complicated Prüfer angle construction.

Unfortunately, it seems to be difficult to apply the Prüfer angle construction to the nonlinear problem. Accordingly, in this paper we use alternative, non-optimal, oscillation counting methods to obtain the required spectral properties of the linear problem, and these are then applied to the nonlinear problem to yield the results mentioned above.

## 1. INTRODUCTION

In this paper we consider the nonlinear boundary value problem consisting of the equation

$$-u'' = f(u), \quad \text{on } (-1, 1), \tag{1.1}$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, together with the multi-point boundary conditions

$$\alpha_0^\pm u(\pm 1) + \beta_0^\pm u'(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm) + \sum_{i=1}^{m^\pm} \beta_i^\pm u'(\eta_i^\pm), \tag{1.2}$$

where  $m^\pm \geq 1$  are integers,  $\alpha_0^\pm, \beta_0^\pm \in \mathbb{R}$ , and, for each  $i = 1, \dots, m^\pm$ , the numbers  $\alpha_i^\pm, \beta_i^\pm \in \mathbb{R}$ , and  $\eta_i^\pm \in [-1, 1]$ , with  $\eta_i^\pm \neq \pm 1$ . We will obtain Rabinowitz-type global bifurcation results, and then use these to obtain ‘nodal’ solutions of the problem. We conclude with a nonresonance result for an inhomogeneous form of the problem.

However, as a preliminary to the discussion of the nonlinear problem, we will discuss the linear eigenvalue problem consisting of the equation

$$-u'' = \lambda u, \quad \text{on } (-1, 1), \tag{1.3}$$

where  $\lambda \in \mathbb{R}$ , together with the boundary conditions (1.2). Naturally, an *eigenvalue* is a number  $\lambda$  for which (1.2)-(1.3), has a non-trivial solution  $u$  (an *eigenfunction*). The *spectrum*,  $\sigma$ , is the set of eigenvalues.

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Throughout the paper we will suppose that the coefficients in (1.2) satisfy the conditions

$$\alpha_0^\pm \geq 0, \quad \alpha_0^\pm + |\beta_0^\pm| > 0, \quad (1.4)$$

$$\pm \beta_0^\pm \geq 0, \quad (1.5)$$

$$\left( \frac{\sum_{i=1}^{m^\pm} |\alpha_i^\pm|}{\alpha_0^\pm} \right)^2 + \left( \frac{\sum_{i=1}^{m^\pm} |\beta_i^\pm|}{\beta_0^\pm} \right)^2 < 1 \quad (1.6)$$

with the convention that if any denominator in (1.6) is zero then the corresponding numerator must also be zero, and the corresponding fraction is omitted from (1.6) (by (1.4), at least one denominator is nonzero in each condition (1.6)).

Although the boundary conditions (1.2) are non-local, for ease of discussion we will say that the condition with superscript  $\pm$  holds ‘at the end point  $\pm 1$ ’, and we will denote these individual conditions by  $(1.2)^\pm$  (and similarly for other conditions such as  $(1.4)^\pm$ ,  $(1.6)^\pm$ , and others below). Also, we will write  $\alpha^\pm := (\alpha_1^\pm, \dots, \alpha_{m^\pm}^\pm) \in \mathbb{R}^{m^\pm}$ , and similarly for  $\beta^\pm, \eta^\pm$ . The notation  $\alpha^\pm = 0$  or  $\beta^\pm = 0$ , will mean the zero vector in  $\mathbb{R}^{m^\pm}$ , as appropriate. When  $\alpha^\pm = \beta^\pm = 0$  the multi-point boundary conditions  $(1.2)^\pm$  reduce to standard, *single-point* conditions at  $x = \pm 1$ , and the overall multi-point problem (1.2)-(1.3) reduces to a standard, linear Sturm-Liouville problem. Thus, we will term the conditions (1.2) *Sturm-Liouville-type* boundary conditions. If  $\beta_0^\pm = 0$  (respectively,  $\alpha_0^\pm = 0$ ) we term the condition  $(1.2)^\pm$  *Dirichlet-type* (respectively, *Neumann-type*). This terminology is motivated by observing that a multi-point Dirichlet-type (respectively Neumann-type) condition reduces to a single-point Dirichlet (respectively Neumann) condition when  $\alpha = 0$  (respectively  $\beta = 0$ ).

Various types of boundary value problems with multi-point boundary conditions, both linear and nonlinear, have been extensively studied recently, see for example, [2–7, 12–16], and the references therein. In this paper we will continue the investigation of the spectral properties of the linear problem (1.2)-(1.3), and then apply these properties to the nonlinear problem (1.1)-(1.2).

**1.1. Previous spectral results.** The spectral properties of the standard linear, single-point, Sturm-Liouville problem are of course well known, but the spectral properties of the above general linear, multi-point problem (1.2)-(1.3) are still being investigated. Indeed, it is only recently that the basic spectral properties of any multi-point problems have been obtained (the multi-point problem is not self-adjoint, so in principle is more ‘difficult’ than the single-point problem, which is self-adjoint). Initially, problems with a single-point condition at one end point and a multi-point condition at the other end point were discussed, essentially using shooting from the single-point end, see for example [2, 7, 12]; the papers [3, 4] also discuss this case, with a variable coefficient function in the differential equation (1.3). Multi-point conditions at both end points are more difficult to deal with. Dirichlet-type conditions, at both end points, were discussed in [5, 13], while Neumann-type conditions were discussed in [14]. The case of a Dirichlet-type condition at one end point and a Neumann-type condition at the other end point was also discussed in [14], where such conditions were termed *mixed*. The full Sturm-Liouville type conditions (1.2) were discussed in [15].

For the problems discussed in the papers [2, 12–15] it was shown that the spectra of these problems have many of the ‘standard’ properties of the spectrum of the usual single-point Sturm-Liouville problem, specifically:

( $\sigma$ -a)  $\sigma$  consists of a strictly increasing sequence of real eigenvalues  $\lambda_k$ ,  $k = 0, 1, \dots$ ;

( $\sigma$ -b)  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ ;

for each  $k \geq 0$ :

( $\sigma$ -c)  $\lambda_k$  has geometric multiplicity 1;

( $\sigma$ -d) the eigenfunctions corresponding to  $\lambda_k$  have an ‘oscillation count’ equal to  $k$ .

In the single-point problem the oscillation count referred to in property ( $\sigma$ -d) is simply the number of interior *nodal* zeros of an eigenfunction in the interval  $(-1, 1)$ . However, in the multi-point problem it was found in [13] and [14] that this method of counting eigenfunction oscillations no longer necessarily yields property ( $\sigma$ -d), and alternative methods were adopted, with different approaches being used for different types of boundary conditions (a more detailed discussion is given in [14, Section 9.4]). The methods of [13, 14] were then unified and extended to the general Sturm-Liouville type boundary conditions in [15], using a Prüfer angle approach to describe the eigenfunction oscillation count, and the above spectral properties ( $\sigma$ -a)-( $\sigma$ -d) were obtained under the above hypotheses (1.4)-(1.6). It was also shown in [15] that these hypotheses were optimal for this result, in the sense that if they do not hold then properties ( $\sigma$ -c) or ( $\sigma$ -d) may not be true – in the former case, there may be eigenvalues with multiplicity 2, while in the latter case, there may be ‘missing’ eigenvalues, that is, there may be values of  $k \geq 0$  for which there is no eigenvalue whose eigenfunctions have the corresponding oscillation count  $k$ .

Unfortunately, although the Prüfer angle approach used in [15] works well for the linear eigenvalue problem, it does not seem to work so well for the nonlinear problem. In particular, Rabinowitz-type global bifurcation results have often been obtained for both single-point and multi-point Sturm-Liouville problems (dating back to Rabinowitz’ seminal paper [8] for the single-point problem and, for example, in [13] and [14] for the Dirichlet-type and Neumann-type problems respectively), and these results have been used to obtain nodal solutions for these problems. Such results rely on the preservation of the nodal properties along the bifurcating continua, and this seems to be difficult to verify for the Prüfer angle oscillation counting method used in [15]. Hence, in this paper we will discuss how the oscillation counting methods of [13] and [14] can be combined to apply to both the linear and the nonlinear Sturm-Liouville multi-point boundary value problems. The results obtained in this manner will not be optimal for the linear eigenvalue problem, but they will yield global bifurcating continua and nodal solutions for the nonlinear problem.

The linear results below will also clarify the use of two different oscillation counting methods for the Dirichlet-type and Neumann-type problems in [13] and [14], and show that these problems can be regarded as extreme ends of a range of Sturm-Liouville boundary conditions, with a gradual switch between the two oscillation counting methods — see Remark 3.16 for a more careful description of this. Of course, the Prüfer angle oscillation count used in [15] also subsumed and generalised the various oscillation counting methods used in [13] and [14] in the Dirichlet-type, Neumann-type (and mixed) cases respectively.

## 2. PRELIMINARIES

**2.1. Some further notation.** Clearly, the eigenvalues  $\lambda_k$  (and other objects to be introduced below) depend on the values of the coefficients  $\alpha_0^\pm$ ,  $\beta_0^\pm$ ,  $\alpha^\pm$ ,  $\beta^\pm$ ,  $\eta^\pm$ , but in general we regard these coefficients as fixed, and omit them from our notation. However, at certain points of the discussion it will be convenient to regard some, or all, of these coefficients as variable, and to indicate the dependence of various functions on these coefficients. To do this concisely we will write:

$$\alpha_0 := (\alpha_0^-, \alpha_0^+) \in \mathbb{R}^2 \quad (\text{for given numbers } \alpha_0^\pm \in \mathbb{R});$$

$$\alpha := (\alpha^-, \alpha^+) \in \mathbb{R}^{m^- + m^+} \quad (\text{for given coefficient vectors } \alpha^\pm \in \mathbb{R}^{m^\pm});$$

and similarly for  $\beta_0$ ,  $\beta$ ,  $\eta$ . We also define  $\mathbf{0} := (0, 0) \in \mathbb{R}^{m^- + m^+}$ . We may then write, for example,  $\lambda_k(\alpha, \beta)$  to indicate the dependence of  $\lambda_k$  on  $(\alpha, \beta)$ .

In addition, when discussing an individual boundary condition (1.2) at  $x = -1$  or  $x = 1$ , it will be convenient to let  $\nu$  denote one of the signs  $\{\pm\}$ , and to use the notation  $(1.2)^\nu$  to refer to the boundary condition (1.2) at the specific end point  $x = \nu 1$  (with the natural interpretation of this); we will use a similar notation for other conditions, such as (1.4)-(1.6), or other conditions below. Also, for  $u \in C^1[-1, 1]$ , the notation  $u(\nu)$  or  $u'(\nu)$  will denote the value of  $u$  or  $u'$  at the end point  $x = \nu 1$ .

For any integer  $n \geq 0$ , let  $C^n[-1, 1]$  denote the usual Banach space of  $n$ -times continuously differentiable functions on  $[-1, 1]$ , with the usual sup-type norm, denoted by  $|\cdot|_n$ . We now define an operator formulation of the differential operator with multi-point boundary conditions. Let

$$\begin{aligned} X &:= \{u \in C^2[-1, 1] : u \text{ satisfies (1.2)}\}, & \|\cdot\|_X &:= |\cdot|_2, \\ Y &:= C^0[-1, 1], & \|\cdot\|_Y &:= |\cdot|_0, \\ \Delta u &:= u'', & u &\in X. \end{aligned}$$

By the definition of the spaces  $X, Y$ , the linear operator  $\Delta : X \rightarrow Y$  is well-defined and bounded, and we can rewrite the eigenvalue problem (1.2)-(1.3) as

$$-\Delta u = \lambda u, \quad u \in X. \quad (2.1)$$

**2.2. Nodal sets.** The nodal/oscillation properties of solutions of nonlinear Sturm-Liouville problems with single-point boundary conditions are usually described in terms of sets of functions  $u \in C^2[-1, 1]$  having a specified number of interior zeros (that is, points  $x \in (-1, 1)$  for which  $u(x) = 0$ ), and satisfying the given boundary conditions, see, for example, [8, Section 2]. However, in the case of multi-point boundary conditions it has also been found useful to count the interior zeros of  $u'$ . Specifically, in [13] and [14] certain sets, denoted  $T_k$  and  $S_k$ , were used to count oscillations in the Dirichlet-type and Neumann-type cases respectively. We recall the definitions of these sets here. For any  $C^1$  function  $u$ , if  $u(x_0) = 0$  then  $x_0$  is a *simple* zero of  $u$  if  $u'(x_0) \neq 0$ .

**Definition 2.1.** For any integer  $k \geq 0$ :

$S_k^+ \subset C^2[-1, 1]$  is the set of functions  $u \in C^2[-1, 1]$  satisfying the conditions:

- S-(a)  $u(\pm 1) \neq 0$  and  $u(-1) > 0$ ;
- S-(b)  $u$  has only simple zeros in  $(-1, 1)$ , and has exactly  $k$  such zeros.

We also define  $S_k^- := -S_k^+$  and  $S_k := S_k^+ \cup S_k^-$ .

$T_k^+ \subset C^2[-1, 1]$  is the set of functions  $u \in C^2[-1, 1]$  satisfying the conditions:

- T-(a)  $u'(\pm 1) \neq 0$  and  $u'(-1) > 0$ ;
- T-(b)  $u'$  has only simple zeros in  $(-1, 1)$ , and has exactly  $k$  such zeros;
- T-(c)  $u$  has a zero strictly between each consecutive zero of  $u'$ .

We also define  $T_k^- := -T_k^+$  and  $T_k := T_k^+ \cup T_k^-$ .

Clearly, the sets  $S_k^\nu$  (respectively  $T_k^\nu$ ),  $k \geq 0$ ,  $\nu \in \{\pm\}$ , are disjoint and open in  $C^2[-1, 1]$ . Another class of nodal spaces, denoted  $P_k = P_k^+ \cup P_k^- \subset \mathbb{R} \times C^2[-1, 1]$ ,  $k \geq 0$ , was defined in [15] to deal with the general Sturm-Liouville type boundary conditions (1.2) above. These sets were defined in terms of the Prüfer angle of solutions of (1.3); the definition is quite long, and the details are not required here, so will be omitted. Suffice it to say that the Prüfer angle approach extends and unifies the two separate approaches adopted in [13] and [14].

**2.3. The nodal properties of the eigenfunctions.** The eigenvalues and eigenfunctions of (1.2)-(1.3) will be denoted by  $\lambda_k, \psi_k, k \geq 0$ . The eigenfunctions will always be normalised so that  $|\psi_k|_0 = 1$ . When  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$  the multi-point boundary conditions (1.2) reduce to the standard (Robin) conditions

$$\alpha_0^\pm u(\pm 1) + \beta_0^\pm u'(\pm 1) = 0, \quad (2.2)$$

and the eigenvalues and eigenfunctions of (1.3), with these boundary conditions, will be denoted by  $\lambda_k^0, \psi_k^0, k \geq 0$ .

The following theorem was proved in [15, Theorem 4.8].

**Theorem 2.2.** *The spectrum  $\sigma$  of  $-\Delta$  consists of a strictly increasing sequence of real eigenvalues  $\lambda_k \geq 0, k = 0, 1, \dots$ , such that  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ , and for each  $k \geq 0$ :*

- (a)  $\lambda_k$  has geometric multiplicity 1;
- (b)  $\lambda_k$  has an eigenfunction  $\psi_k$  such that  $(\lambda_k, \psi_k) \in P_k$ .

In the Neumann-type case  $\lambda_0 = 0$ , while if  $\alpha_0^- + \alpha_0^+ > 0$  then  $\lambda_0 > 0$ .

*Proof (sketch).* For any  $k \geq 0$  the multi-point eigenvalue  $\lambda_k(\alpha, \beta)$  and eigenfunction  $\psi_k(\alpha, \beta)$  were constructed in [15] by continuation from the single-point eigenvalue  $\lambda_k^0$  and eigenfunction  $\psi_k^0$ . In essence, this continuation construction showed that the mappings

$$t \rightarrow \psi_k[t] := \psi_k(t\alpha, t\beta) : [0, 1] \rightarrow C^2[-1, 1], \quad t \rightarrow \lambda_k[t] := \lambda_k(t\alpha, t\beta) : [0, 1] \rightarrow \mathbb{R}, \quad (2.3)$$

are well-defined and continuous, and when  $t = 0$  they satisfy

$$\psi_k[0] = \psi_k(\mathbf{0}, \mathbf{0}) = \psi_k^0, \quad \lambda_k[0] = \lambda_k(\mathbf{0}, \mathbf{0}) = \lambda_k^0. \quad (2.4)$$

The multi-point eigenvalue and eigenfunction,  $\lambda_k, \psi_k$ , are then obtained by setting  $t = 1$ . The properties of  $\lambda_k, \psi_k$  can be derived from the corresponding properties of the single-point eigenvalue and eigenfunction,  $\lambda_k^0, \psi_k^0$  (which are obtained from standard Sturm-Liouville theory) by showing that they are preserved during this continuation process, as  $t$  varies from 0 to 1.  $\square$

**Remark 2.3.** The sign condition (1.5) ensures that  $\lambda_0^0 > 0$  (except in the Neumann-type case, when  $\lambda_0^0 = 0$ ), and this positivity is preserved in the continuation. It is shown in [15] that if (1.5) does not hold then negative eigenvalues may exist, and these may have geometric multiplicity 2 (of course, this cannot happen in the standard, single-point problem). It is also shown in [15] that if (1.6) does not hold then there may be values of  $k$  for which there is no eigenvalue/eigenfunction pair  $(\lambda_k, \psi_k) \in P_k$ . Hence, the ‘standard’ spectral properties described in Theorem 2.2 may not hold if either (1.5) or (1.6) are not satisfied.

### 3. NODAL PROPERTIES OF EIGENFUNCTIONS

We will now ascertain the nodal properties of the multi-point eigenfunctions described in Theorem 2.2, in terms of the nodal sets  $T_k$  and  $S_k$  (and a further class of such sets to be introduced below), instead of the sets  $P_k$  used in [15]. We begin with some preliminary results, which will form the basis of the discussion of nodal properties. We first note that it can easily be shown that for any solution  $(\lambda, u)$ ,  $\lambda > 0$ , of (1.3) we have the elementary ‘energy’ equalities:

$$\lambda u(x)^2 + u'(x)^2 \equiv \lambda |u|_0^2 = |u'|_0^2, \quad x \in [-1, 1]. \quad (3.1)$$

**Lemma 3.1.** *Suppose that  $(\lambda, u)$ ,  $\lambda > 0$ ,  $u \not\equiv 0$ , satisfies (1.3) and  $(1.2)^\nu$ , for some  $\nu \in \{\pm\}$ . Then:*

(a) if

$$\alpha_0^\nu > \sum_{i=1}^{m^\nu} |\alpha_i^\nu| + \lambda^{1/2} \sum_{i=1}^{m^\nu} |\beta_i^\nu|, \quad (3.2)$$

then  $u'(\nu) \neq 0$ ;

(b) if

$$\beta_0^\nu > \frac{1}{\lambda^{1/2}} \sum_{i=1}^{m^\nu} |\alpha_i^\nu| + \sum_{i=1}^{m^\nu} |\beta_i^\nu|, \quad (3.3)$$

then  $u(\nu) \neq 0$ .

*Proof.* Suppose that  $u'(\nu) = 0$ . Then it follows from  $(1.2)^\nu$  and (3.1) that

$$\alpha_0^\nu |u|_0 \leq |u|_0 \sum_{i=1}^{m^\nu} |\alpha_i^\nu| + |u'|_0 \sum_{i=1}^{m^\nu} |\beta_i^\nu| \leq |u|_0 \sum_{i=1}^{m^\nu} |\alpha_i^\nu| + \lambda^{1/2} |u|_0 \sum_{i=1}^{m^\nu} |\beta_i^\nu|,$$

which contradicts (3.2), and so proves part (a). The proof of part (b) is similar.  $\square$

**3.1. The case of one multi-point boundary condition.** We begin the discussion of the nodal properties of the eigenfunctions by considering the simpler case where we only have one multi-point boundary condition. Specifically, in this section we suppose that in the boundary condition  $(1.2)^-$  we have

$$\alpha^- = \beta^- = 0, \quad (3.4)$$

that is, at  $x = -1$  the boundary condition  $(1.2)^-$  simply reduces to the Robin condition  $(2.2)^-$ , while we retain the multi-point boundary condition at  $x = 1$ . The case where we only have a multi-point condition at  $x = -1$  is entirely similar.

We first observe that if  $u$  is a non-trivial solution of  $(2.2)^-$ , (1.3), and if  $\alpha_0^- \beta_0^- \neq 0$  then  $u(-1)u'(-1) \neq 0$ , which is consistent with  $u$  belonging to either of the nodal sets  $S_k, T_k$ , for some  $k \geq 0$ . On the other hand, if  $\alpha_0^- = 0$  (respectively  $\beta_0^- = 0$ ) then  $u'(-1) = 0$  (respectively  $u(-1) = 0$ ), in which case  $u$  cannot belong to any set  $T_k$  (respectively  $S_k$ ),  $k \geq 0$ , so one of the classes of nodal sets  $S_k, T_k$  is of no use in this case. However, this problem is easily remedied by simply redefining the sets  $S_k, T_k$  to only include functions satisfying the boundary condition  $(2.2)^-$ . Such redefined sets would not be open in  $C^2[-1, 1]$ , but would be open in the subset of  $C^2[-1, 1]$  consisting of functions satisfying  $(2.2)^-$ , which suffices for the arguments below (in this section). We will not mention this special case again, but in any of the following results in this section we will implicitly suppose that we are using the redefined sets in this case.

Next, we introduce some further definitions. We denote the single-point eigenvalues and eigenfunctions of (1.3), with the boundary condition  $(2.2)^-$  at  $x = -1$ , together with Dirichlet or Neumann boundary conditions at  $x = 1$ , by:  $\lambda_k^{RD}, \psi_k^{RD}, \lambda_k^{RN}, \psi_k^{RN}, k \geq 0$ . It can be verified that if  $(\lambda, u)$ ,  $\lambda > 0$ , satisfies (1.3),  $(2.2)^-$ , then, for each integer  $k \geq 0$ ,

$$u \in T_{k+1} \implies \lambda_k^{RN} < \lambda < \lambda_{k+1}^{RN}, \quad (3.5)$$

$$u \in S_k \implies \lambda_{k-1}^{RD} < \lambda < \lambda_k^{RD} \quad (3.6)$$

(we define  $\lambda_{-1}^{RD} := 0$ ).

**Theorem 3.2.** *Suppose that (3.4) holds. Then, for any integer  $k_0 \geq 0$ :*

(a) if  $(3.2)^+$  holds for  $\lambda = \lambda_{k_0}^{RN}$  then

$$k \leq k_0 - 1 \implies \psi_k \in T_{k+1} \quad \text{and} \quad \lambda_k^{RN} < \lambda_k < \lambda_{k+1}^{RN}; \quad (3.7)$$

(b) if  $(3.3)^+$  holds for  $\lambda = \lambda_{k_0}^{RD}$  then

$$k \geq k_0 + 1 \implies \psi_k \in S_k \quad \text{and} \quad \lambda_{k-1}^{RD} < \lambda_k < \lambda_k^{RD}. \quad (3.8)$$

If  $(3.3)^+$  holds for  $\lambda = \lambda_0^{RD}$  then (3.8) also holds for  $k = 0$ , so  $\psi_k \in S_k$ , for all  $k \geq 0$ .

*Proof.* The proof relies on the continuation construction of the eigenvalues and eigenfunctions, as described in the above sketch of the proof of Theorem 2.2. In particular, we use the mappings (2.3), with the properties (2.4).

We first assume that  $\alpha_0^+ \neq 0$  and  $\beta_0^+ \neq 0$ . Now, for any  $k \geq 0$ , the eigenfunction  $\psi_k^0$  satisfies the boundary condition  $(2.2)^+$  at  $x = 1$ , so by (1.4) and (1.5),

$$\text{sgn } \psi_k^0(1) = -\text{sgn } \frac{d\psi_k^0}{dx}(1) \neq 0, \quad (3.9)$$

from which the following additional nodal and eigenvalue interlacing properties can be obtained,

$$\psi_k^0 \in S_k \cap T_{k+1}, \quad \lambda_k^{RN} < \lambda_k^0 < \lambda_k^{RD} < \lambda_{k+1}^{RN}, \quad k \geq 0. \quad (3.10)$$

(a) Suppose that  $k \leq k_0 - 1$ . It follows from (3.5) and part (a) of Lemma 3.1 that, for  $t \in [0, 1]$ ,

$$\psi_k[t] \in T_{k+1} \implies \lambda_k^{RN} < \lambda_k[t] < \lambda_{k+1}^{RN} \leq \lambda_{k_0}^{RN}, \quad (3.11)$$

$$\lambda_k[t] \leq \lambda_{k_0}^{RN} \implies \psi'_k[t](1) \neq 0 \implies \psi_k[t] \notin \partial T_{k+1}. \quad (3.12)$$

Also, (3.10) shows that the left hand sides of the implications (3.11)-(3.12) hold when  $t = 0$ , so by continuity the right hand sides hold for all  $t \in [0, 1]$ , and putting  $t = 1$  yields (3.7).

(b) The proof of (3.8) is similar, using (3.6) and part (b) of Lemma 3.1. The analogues of the implications (3.11)-(3.12) in this case are:

$$\psi_k[t] \in S_k \implies \lambda_{k_0}^{RD} \leq \lambda_{k-1}^{RD} < \lambda_k[t] < \lambda_k^{RD}, \quad (3.13)$$

$$\lambda_k[t] \geq \lambda_{k_0}^{RD} \implies \psi_k[t](1) \neq 0 \implies \psi_k[t] \notin \partial S_k, \quad (3.14)$$

and (3.8) now follows from these implications, as before.

The final result follows from a similar argument, but the lower bound  $0 < \lambda_0[t]$  is now trivial, so we only need to prevent  $\lambda_0[t]$  crossing  $\lambda_0^{RD}$ , which follows from the assumption that  $(3.3)^+$  holds for  $\lambda = \lambda_0^{RD}$  and part (b) of Lemma 3.1.

The cases  $\alpha_0^+ = 0$ ,  $\beta_0^+ > 0$ , and  $\alpha_0^+ > 0$ ,  $\beta_0^+ = 0$ , may be proved similarly, but a generalisation of these cases is stated, and proved, in the following corollary, so we omit any further discussion of these cases here.  $\square$

**Corollary 3.3.** (a) If  $\alpha_0^+ > 0$ ,  $\beta^+ = 0$ , then (3.7) holds for all  $k \geq 0$ .

(b) If  $\alpha^+ = 0$ ,  $\beta_0^+ > 0$ , then (3.8) holds for all  $k \geq 0$ .

(c) If  $\beta_0^+ \neq 0$ , (3.8) holds for all sufficiently large  $k$ .

*Proof.* In case (a),  $(3.2)^+$  holds for all  $\lambda > 0$ , so we follow the proof of part (a) of Theorem 3.2. In case (b) (respectively, case (c)),  $(3.3)^+$  holds for all  $\lambda > 0$  (respectively, for sufficiently large  $\lambda > 0$ ), so we follow the proof of part (b) of Theorem 3.2.  $\square$

Theorem 3.2 deals with ‘most’ eigenvalues, but there can be an arbitrarily large ‘gap’ or range of ‘intermediate’ eigenvalues for which neither of the hypotheses  $(3.2)^+$  or  $(3.3)^+$



hold, so are not covered by this theorem. For example, if

$$\alpha_0^+ = \sqrt{2}, \quad \sum_{i=1}^{m^+} |\alpha_i^+| = 1, \quad \beta_0^+ = \epsilon\sqrt{2.1}, \quad \sum_{i=1}^{m^+} |\beta_i^+| = \epsilon,$$

then (1.6) holds, but

$$(3.2)^+ \implies \lambda < \frac{\sqrt{2}-1}{\epsilon} \approx \frac{0.41}{\epsilon}, \quad (3.3)^+ \implies \lambda > \frac{1}{\epsilon(\sqrt{2.1}-1)} \approx \frac{2.23}{\epsilon},$$

so it follows from the Sturm comparison theorem that if  $\epsilon$  is sufficiently small then an arbitrarily large number of eigenvalues do not satisfy either  $(3.2)^+$  or  $(3.3)^+$ .

We can remove this gap by strengthening condition (1.6) somewhat. Specifically, if we replace (1.6) with the condition

$$\frac{\sum_{i=1}^{m^\pm} |\alpha_i^\pm|}{\alpha_0^\pm} + \frac{\sum_{i=1}^{m^\pm} |\beta_i^\pm|}{\beta_0^\pm} < 1 \quad (3.15)$$

(in this section the condition  $(3.15)^-$  holds trivially, and is irrelevant, but will be used in the next section). The inequalities  $(3.2)^\nu$  and  $(3.3)^\nu$  are related to the condition  $(3.15)^\nu$ . To clarify this relationship, let

$$J^\pm := \left( \frac{\alpha_0^\pm}{\beta_0^\pm} \right)^2$$

(if, for either  $\nu \in \{\pm\}$ , we have  $\beta_0^\nu = 0$  then we set  $J^\nu := \infty$ , and the results below hold, with the natural interpretation of this). We now have the following corollary of Lemma 3.1 which shows that if  $(3.15)^\nu$  holds then there is no gap between the values of  $\lambda$  for which  $(3.2)^\nu$  and  $(3.3)^\nu$  hold.

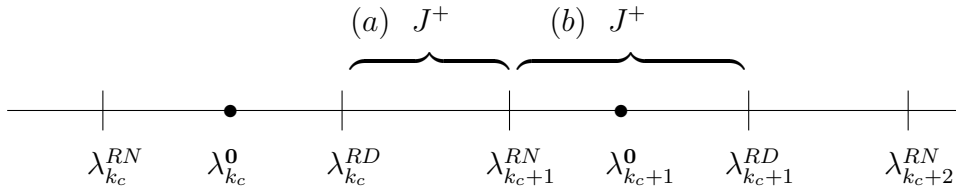
**Corollary 3.4.** *Suppose that  $(\lambda, u)$ ,  $\lambda > 0$ ,  $u \not\equiv 0$ , satisfies (1.3) and, for some  $\nu \in \{\pm\}$ ,  $(1.2)^\nu$  and  $(3.15)^\nu$  hold. Then:*

- (a)  $\lambda \leq J^\nu \implies (3.2)^\nu \text{ holds} \implies u'(\nu) \neq 0;$
- (b)  $\lambda \geq J^\nu \implies (3.3)^\nu \text{ holds} \implies u(\nu) \neq 0.$

Next, it is easy to verify that  $\lambda_k^{RN} < \lambda_k^{RD} < \lambda_{k+1}^{RN}$ , so there exists a unique integer  $k_c \geq -1$  such that

$$\lambda_{k_c}^{RD} < J^+ \leq \lambda_{k_c+1}^{RD} \quad (3.16)$$

(if  $J^+ = 0$ , we set  $k_c := -1$ ). These definitions and eigenvalue interlacing properties are illustrated in Fig. 1.



**Figure 1:** Eigenvalue interlacing and the definition of  $J^+$  (cases (a) and (b) refer to the hypotheses in Theorem 3.6).

Combining this definition with Corollary 3.4, we see that

$$\lambda \leq \lambda_{k_c}^{RD} \implies (3.2)^+ \text{ holds}, \quad \lambda \geq \lambda_{k_c+1}^{RD} \implies (3.3)^+ \text{ holds},$$

and combining all this with Theorem 3.2 yields the following result.

**Theorem 3.5.** *Suppose that (3.4) and (3.15)<sup>+</sup> hold. Then, for any integer  $k \geq 0$ :*

$$k \leq k_c - 1 \implies \psi_k \in T_{k+1} \quad \text{and} \quad \lambda_k^{RN} < \lambda_k < \lambda_{k+1}^{RN}; \quad (3.17)$$

$$k \geq k_c + 2 \implies \psi_k \in S_k \quad \text{and} \quad \lambda_{k-1}^{RD} < \lambda_k < \lambda_k^{RD}. \quad (3.18)$$

If  $k_c = -1$ , that is, if  $J^+ \leq \lambda_0^{RD}$ , then (3.18) also holds for  $k = 0$ , so  $\psi_k \in S_k$ , for all  $k \geq 0$ .

Theorem 3.5 has dealt with all the eigenvalues in  $\sigma$  except those with index  $k = k_c$  or  $k = k_c + 1$ . We deal with these in the next theorem.

**Theorem 3.6.** *Suppose that (3.4) and (3.15)<sup>+</sup> hold. Suppose, in addition, that one of the following conditions holds:*

$$(a) \quad \lambda_{k_c}^{RD} < J^+ < \lambda_{k_c+1}^{RN} \quad \text{and either}$$

$$(i) \quad \lambda = \lambda_{k_c}^{RD} \text{ satisfies (3.3)}^+,$$

$$(ii) \quad \lambda = \lambda_{k_c+1}^{RN} \text{ satisfies (3.2)}^+;$$

$$(b) \quad \lambda_{k_c+1}^{RN} \leq J^+ \leq \lambda_{k_c+1}^{RD}.$$

Then (3.17) holds when  $k = k_c$  and (3.18) holds when  $k = k_c + 1$ .

*Proof.* The proof is similar to the proof of Theorem 3.2. Heuristically, we can describe the argument as follows (again, see Fig. 1). Combining the definition of  $k_c$  with Corollary 3.4 shows that during the continuation process the eigenvalues  $\lambda_k[t]$ ,  $t \in [0, 1]$ ,  $k = k_c$  or  $k = k_c + 1$ , cannot cross either  $\lambda_{k_c}^{RN}$  or  $\lambda_{k_c+1}^{RD}$ . In addition, each set of hypotheses in the theorem ensures that these eigenvalues also cannot cross one or other of  $\lambda_{k_c}^{RD}$  or  $\lambda_{k_c+1}^{RN}$ . Combining these bounds on the eigenvalues yields the result.  $\square$

**Remark 3.7.** By Theorem 2.2 above (proved in [15, Theorem 4.8]), the basic hypotheses (1.6)<sup>±</sup> (together with the other conditions in Section 1) are sufficiently strong to imply that for every integer  $k \geq 0$  there is exactly one eigenvalue whose eigenfunctions lie in the nodal set  $P_k$  (with either one or two multi-point boundary conditions). It is also shown in [15, Section 4.5] that if (1.6) is weakened by replacing 1 on the right hand side with  $1 + \epsilon$ , for arbitrarily small  $\epsilon > 0$ , then this is no longer true. However, even in the case of one multi-point condition, as considered in this section, (1.6)<sup>+</sup> does not seem to be sufficiently strong to ensure that the nodal properties of all the eigenfunctions can always be described in terms of the nodal sets  $S_k, T_k$ , for all  $\alpha_0^+, \beta_0^+, \alpha^+, \beta^+$ , satisfying (1.6)<sup>+</sup>. On the other hand, condition (3.15)<sup>+</sup> does ensure this for all the eigenvalues, except those considered in part (a) of Theorem 3.6. In this case the additional conditions (i) or (ii) were imposed there to deal with the eigenvalues  $\lambda_{k_c}$  and  $\lambda_{k_c+1}$ . These conditions represent a slight strengthening of (3.15)<sup>+</sup> in that, in general, they require  $\sum_{i=1}^{m^+} |\alpha_i^+|$  and  $\sum_{i=1}^{m^+} |\beta_i^+|$  to be smaller than required by (3.15)<sup>+</sup>. We also note that  $J^+$  and  $k_c$  depend only on  $\alpha_0^+, \beta_0^+$ , so they do not ‘see’ how small  $\alpha^+, \beta^+$  are, and we expect to obtain stronger results when these coefficient vectors are small (when they are zero the problem reduces to the standard Sturm-Liouville problem).

**3.2. The case of two multi-point boundary conditions.** We now discuss the nodal properties of the multi-point eigenfunctions with two multi-point boundary conditions. We first need some more definitions. The standard single-point eigenvalues and eigenfunctions of (1.3), with Dirichlet, Neumann and mixed (i.e., Dirichlet at one end point and Neumann at the other end) boundary conditions, will be denoted by  $\lambda_k^D, \psi_k^D, \lambda_k^N, \psi_k^N, \lambda_k^M, \psi_k^M$ . It

can be verified that if  $(\lambda, u)$ ,  $\lambda > 0$ , satisfies (1.3), then, for each integer  $k \geq 0$ ,

$$u \in T_{k+1} \implies \lambda_k^N < \lambda < \lambda_{k+2}^N, \quad (3.19)$$

$$u \in S_k \implies \lambda_{k-2}^D < \lambda < \lambda_k^D \quad (3.20)$$

(we define  $\lambda_{-2}^D := 0$ ,  $\lambda_{-1}^D := 0$ ). We now have an analogue of Theorem 3.2.

**Theorem 3.8.** *For any integer  $k_0 \geq 0$ :*

(a) *if (3.2)<sup>±</sup> holds for  $\lambda = \lambda_{k_0}^N$  then*

$$k \leq k_0 - 2 \implies \psi_k \in T_{k+1} \quad \text{and} \quad \lambda_k^N < \lambda_k < \lambda_{k+2}^N; \quad (3.21)$$

(b) *if (3.3)<sup>±</sup> holds for  $\lambda = \lambda_{k_0}^D$  then*

$$k \geq k_0 + 2 \implies \psi_k \in S_k \quad \text{and} \quad \lambda_{k-2}^D < \lambda_k < \lambda_k^D. \quad (3.22)$$

*Proof.* The proof again relies on the continuation construction of the eigenvalues and eigenfunctions. We first note that if  $\alpha_0^\nu \neq 0$  and  $\beta_0^\nu \neq 0$ , for some  $\nu \in \{\pm\}$ , then, by (1.4), (1.5) and (2.2), the eigenfunction  $\psi_k^0$  satisfies

$$\text{sgn } \psi_k^0(\nu) = -\nu \text{sgn } \frac{d\psi_k^0}{dx}(\nu) \neq 0, \quad (3.23)$$

from which the following nodal and eigenvalue interlacing properties can be obtained,

$$\psi_k^N, \psi_k^0 \in S_k, \quad \psi_k^0, \psi_k^D \in T_{k+1}, \quad \lambda_{k-1}^D = \lambda_k^N < \lambda_k^0, \quad \lambda_k^M < \lambda_k^D = \lambda_{k+1}^N, \quad k \geq 0. \quad (3.24)$$

Also, if  $\alpha_0^\nu = 0$  and  $\beta_0^{-\nu} = 0$ , for some  $\nu \in \{\pm\}$ , then  $\lambda_k^0$  is a mixed eigenvalue, so the properties (3.24) again hold. We suppose for now that either of these cases hold, and so the properties (3.24) hold; the cases  $\alpha_0^\pm = 0$  or  $\beta_0^\pm = 0$ , when this is not so, will be considered below.

(a) Suppose that  $k \leq k_0 - 2$ . Then, by (3.19) and part (a) of Lemma 3.1, for  $t \in (0, 1]$ ,

$$\psi_k[t] \in T_{k+1} \implies \lambda_k^N < \lambda_k[t] < \lambda_{k+2}^N \leq \lambda_{k_0}^N, \quad (3.25)$$

$$\lambda_k[t] \leq \lambda_{k_0}^N \implies \psi'_k[t](\pm 1) \neq 0 \implies \psi_k[t] \notin \partial T_{k+1}. \quad (3.26)$$

Also, (3.24) shows that the left hand sides of the implications (3.25)-(3.26) hold when  $t = 0$ , so by continuity the right hand sides hold for all  $t \in [0, 1]$ , and putting  $t = 1$  yields (3.21).

(b) If  $k \geq k_0 + 2$  then the proof of (3.22) is similar. By (3.20), the analogues of the implications (3.25)-(3.26) in this case are:

$$\psi_k[t] \in S_k \implies \lambda_{k_0}^D \leq \lambda_{k-2}^D < \lambda_k[t] < \lambda_k^D, \quad (3.27)$$

$$\lambda_k[t] \geq \lambda_{k_0}^D \implies \psi_k[t](\pm 1) \neq 0 \implies \psi_k[t] \notin \partial S_k, \quad (3.28)$$

and (3.22) now follows from these implications, as before.

Finally, suppose that  $\beta_0^\pm = 0$ . Then  $\lambda_k^0$  is now a Dirichlet eigenvalue, that is,  $\lambda_k^0 = \lambda_k^D$ , so although the statements in (3.24) regarding  $\lambda_k^0$  are not all correct in this case, the properties of the Dirichlet and Neumann eigenvalues and eigenfunctions are still correct, and these suffice to show that the left hand sides of the implications (3.25)-(3.26) hold when  $t = 0$ . Hence, we can again obtain (3.21) by continuation. Similarly, if  $\alpha_0^\pm = 0$  then  $\lambda_k^0$  is a Neumann eigenvalue and we can again obtain (3.22).  $\square$

We now have the following analogue of Corollary 3.3, with a similar proof, based on the proof of Theorem 3.8.

**Corollary 3.9.** (a) If  $\alpha_0^\pm > 0$ ,  $\beta^\pm = 0$  then (3.21) holds for all  $k \geq 0$ .  
 (b) If  $\alpha^\pm = 0$ ,  $\beta_0^\pm > 0$  then (3.22) holds for all  $k \geq 0$ .  
 (c) If  $\beta_0^\pm > 0$  then (3.22) holds for all sufficiently large  $k$ .

**Remark 3.10.** Corollary 3.9 recovers the nodal properties found in [13, Theorem 5.1] and [14, Theorem 5.1], in the Dirichlet-type and Neumann-type cases respectively. In fact, Corollary 3.9 obtains slightly more since [13] assumes that  $\alpha_0^\pm = 0$ , and [14] assumes that  $\beta_0^\pm = 0$ , whereas Corollary 3.9 allows for both  $\alpha_0^\pm \neq 0$  and  $\beta_0^\pm \neq 0$  simultaneously, although such cases could probably have been tackled using the methods of these previous papers.

As in Section 3.1, there is a range of intermediate eigenvalues not covered by Theorem 3.8. We can start to deal with these using the conditions  $(3.15)^\pm$ . By analogy with the definition of  $k_c$  in (3.16), we define

$$J^{\min} := \min\{J^\pm\}, \quad J^{\max} := \max\{J^\pm\},$$

$$k_T := \max\{k : \lambda_k^N \leq J^{\min}\}, \quad k_S := \min\{k : \lambda_k^D \geq J^{\max}\}. \quad (3.29)$$

Combining these definitions with Corollary 3.4 shows that

$$\lambda \leq \lambda_{k_T}^N \implies (3.2)^\pm \text{ hold}, \quad \lambda \geq \lambda_{k_S}^D \implies (3.3)^\pm \text{ hold},$$

and combining all this with Theorem 3.8 yields the following analogue of Theorem 3.5.

**Theorem 3.11.** Suppose that  $(3.15)^\pm$  hold. Then, for any integer  $k \geq 0$ :

$$k \leq k_T - 2 \implies \psi_k \in T_{k+1} \quad \text{and} \quad \lambda_k^N < \lambda_k < \lambda_{k+2}^N; \quad (3.30)$$

$$k \geq k_S + 2 \implies \psi_k \in S_k \quad \text{and} \quad \lambda_{k-2}^D < \lambda_k < \lambda_k^D. \quad (3.31)$$

**Remark 3.12.** Unfortunately, as we saw in Section 3.1 when dealing with the single multi-point boundary condition case, there is again a gap between the ranges of eigenvalues considered in Theorems 3.8 or 3.11. In Section 3.1 this gap was due to a gap between the values of  $\lambda$  at which the conditions  $(3.2)^+$  and  $(3.3)^+$  hold, and could be eliminated by slightly strengthening the condition  $(1.6)^+$ . However, when we have two multi-point boundary conditions there is also a gap caused by the differences between the ranges of the values of  $\lambda$  at which the ‘switchover’ between the conditions  $(3.2)^\pm$  and  $(3.2)^\pm$  occurs at the two end points  $\pm 1$ . This gap causes a significant additional difficulty in describing the nodal properties, which we now describe.

The proof of Theorem 3.8 used the fact that in the continuation process:

- if  $\lambda_k[t] \leq \lambda_{k_0}^N$  then both conditions  $(3.2)^\pm$  hold, so zeros of  $\psi'_k[t]$  cannot cross either of the end points  $\pm 1$ , while zeros of  $\psi_k[t]$  might cross both;
- if  $\lambda_k[t] \geq \lambda_{k_0}^D$  then both conditions  $(3.3)^\pm$  hold, so zeros of  $\psi_k[t]$  cannot cross either of the end points  $\pm 1$ , while zeros of  $\psi'_k[t]$  might cross both.

In the intermediate range, when  $\lambda_{k_0}^N < \lambda_k[t] < \lambda_{k_0}^D$  (which was not considered in Theorem 3.11), even if  $(3.15)^\pm$  both hold it might be the case that  $(3.2)^\nu$  holds at one end point  $\nu$ , while  $(3.3)^{-\nu}$  holds at the other end point  $-\nu$ . Hence, during the continuation process, a zero of  $\psi_k[t]$  might cross one end point, while a zero of  $\psi'_k[t]$  might cross the other end point, so that neither zeros of  $\psi_k[t]$  nor of  $\psi'_k[t]$  are preserved during the continuation. This renders both the classes of nodal sets  $S_k$  and  $T_k$ ,  $k \geq 0$ , unsuitable for dealing with this intermediate case and necessitates the introduction of another class of nodal sets. We will discuss this in the following subsection.

3.2.1. *Intermediate eigenvalues when there are two multi-point BCs.* For simplicity, throughout this subsection we will suppose that  $(3.15)^\pm$  hold, and

$$J^+ < J^-;$$

the case  $J^- < J^+$  is similar (the case  $J^- = J^+$  is irrelevant in this section). It follows from this, together with Corollary 3.4, that if  $(\lambda, u)$ ,  $\lambda > 0$ , satisfies, (1.2) (1.3), then

$$J^+ \leq \lambda \leq J^- \implies u'(-1) \neq 0, \quad u(1) \neq 0. \quad (3.32)$$

To utilize (3.32) we now introduce the following nodal sets.

**Definition 3.13.** For any integer  $k \geq -1$ ,  $R_k^+ \subset C^2[-1, 1]$  is the set of functions  $u \in C^2[-1, 1]$  satisfying the conditions:

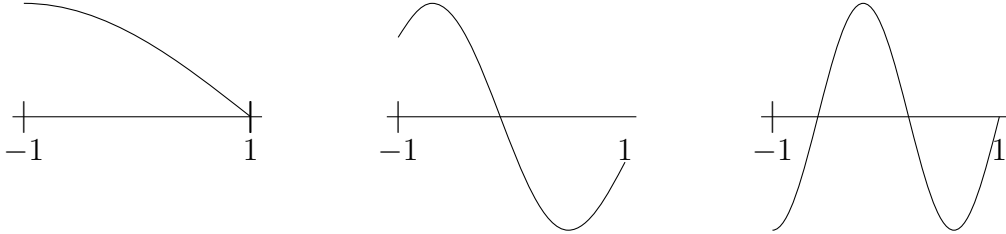
- R-(a)  $u'(-1) > 0$ , and  $u(1) > 0$  iff  $k$  is even,  $u(1) < 0$  iff  $k$  is odd;
- R-(b)  $u$  has only simple zeros in  $(-1, 1)$ , and has either  $k$  or  $k + 1$  such zeros.

We also define  $R_k^- := -R_k^+$  and  $R_k := R_k^+ \cup R_k^-$ .

These sets were defined in [14, Section 9], while a motivation for their somewhat strange definition was discussed in [14, Section 9.4]. Suffice it to say here that, combined with (3.32), they will enable us to extend the above results to (most of) the intermediate eigenvalues. It was shown in [14, Lemma 9.2] that the sets  $R_k^\nu$ ,  $k \geq -1$ ,  $\nu \in \{\pm\}$ , are disjoint and open. In addition, if  $(\lambda, u)$ ,  $\lambda > 0$ , is an arbitrary solution of (1.3), then for any  $k \geq 0$

$$u \in R_k \implies \lambda_{k-1}^M < \lambda < \lambda_{k+1}^M \quad (3.33)$$

(with  $\lambda_{-1}^M := 0$ ). This is analogous to (3.19), (3.20), and is illustrated in Fig. 2, for the case  $k = 1$ . The proof is elementary, based on the definitions, and the properties of the sine function.



$$\lambda = \lambda_0^M, u = \psi_0^M \in \partial R_1 \quad \lambda = \lambda_1^0, u = \psi_1^0 \in R_1 \quad \lambda = \lambda_2^M, u = \psi_2^M \in \partial R_1$$

**Figure 2:** Eigenfunctions corresponding to various eigenvalues.

The relevant interlacing properties are in (3.24). We also let

$$k_{T,M} := \max\{k : \lambda_k^M \leq J^{\min}\}, \quad k_{S,M} := \min\{k : \lambda_k^M \geq J^{\max}\}. \quad (3.34)$$

Comparing (3.34) with the definitions of  $k_T$ ,  $k_S$ , in (3.29), and recalling (3.24), we see that

$$k_T - 1 \leq k_{T,M} \leq k_T, \quad k_S \leq k_{S,M} \leq k_S + 1. \quad (3.35)$$

We now extend Theorem 3.11 to most of the eigenvalues omitted from that result.

**Theorem 3.14.** Suppose that  $(3.15)^\pm$  hold. Then, for any integer  $k \geq 0$ ,

$$k_{T,M} + 1 \leq k \leq k_{S,M} - 1 \implies \psi_k \in R_k \quad \text{and} \quad \lambda_{k-1}^M < \lambda_k < \lambda_{k+1}^M. \quad (3.36)$$

*Proof.* The proof is similar to the proof of Theorem 3.8. In this case, by (3.24), (3.32) and (3.33), the analogues of the implications (3.25)-(3.26) and (3.27)-(3.28) are

$$\begin{aligned} \psi_k[t] \in R_k &\implies J^+ \leq \lambda_{k_{T,M}}^M \leq \lambda_{k-1}^M < \lambda_k[t] < \lambda_{k+1}^M \leq \lambda_{k_{S,M}}^M \leq J^-, \\ J^+ \leq \lambda_k[t] \leq J^- &\implies \psi'_k[t](-1) \psi_k[t](1) \neq 0 \implies \psi_k[t] \notin \partial R_k, \end{aligned}$$

and (3.24) again shows that the left hand sides of these implications hold when  $t = 0$  (a slight extension shows that  $\psi_k^0 \in R_k$ ), so (3.36) now follows by continuity, as before.  $\square$

There is still a small number (at most 4) of eigenvalues that are not covered by Theorems 3.11 and 3.14, viz., those with indices

$$k_T - 1 \leq k \leq k_{T,M}, \quad k_{S,M} \leq k \leq k_S + 1$$

(by (3.35), each of these pairs of inequalities corresponds to either 1 or 2 eigenvalues). In a similar manner to the situation discussed in Remark 3.7 (in the single multi-point BC case), the hypothesis  $(3.15)^\pm$  does not seem to be sufficiently strong to deal with these eigenvalues. However, as in Theorem 3.6, a slight strengthening of  $(3.15)^\pm$  enables us to deal with these eigenvalues. We can immediately derive one such result from Theorem 3.8.

**Theorem 3.15.** *Suppose that  $(3.15)^\pm$  hold. Then:*

- (a) *if  $(3.2)^\pm$  holds for  $\lambda = \lambda_{k_0}^N$ , with  $k_0 = k_{T,M} + 2$  then (3.21) holds for  $k \leq k_{T,M}$ ;*
- (b) *if  $(3.3)^\pm$  holds for  $\lambda = \lambda_{k_0}^D$ , with  $k_0 = k_{S,M} - 2$  then (3.22) holds for  $k \geq k_{S,M}$ .*

**Remark 3.16.** The Dirichlet-type problem ( $\beta_0^\pm = 0$ ) was considered in [13] using the sets  $T_k$ , while the Neumann-type problem ( $\alpha_0^\pm = 0$ ) was considered in [14] using the sets  $S_k$ . Looked at in isolation this use of two different types of nodal sets for the two cases seems slightly strange. However, the above results now show that these cases are simply extreme ends of a range of cases, in the following (somewhat heuristic) sense:

- $\alpha_0^\pm = 0 \implies J^\pm = 0$ : the nodal properties can be described using only the sets  $S_k$ ;
- $\beta_0^\pm = 0 \implies J^\pm = \infty$ : the nodal properties can be described using only the sets  $T_k$ ;
- $0 < J^\pm < \infty$ : the nodal properties are described using a mixture of the sets  $S_k$  and  $T_k$  (and  $R_k$  in an intermediate range), and as  $J^\pm$  varies from 0 to  $\infty$ , the intermediate range (interpreted broadly) between the sets  $S_k$  and  $T_k$  (either  $k_c$  in Section 3.1, or the range between  $k_T$  and  $k_S$  in Section 3.2) varies from  $\infty$  to 0.

#### 4. NONLINEAR PROBLEMS

Using the above discussion of the linear problem we will now proceed to consider some nonlinear problems, including the problem (1.1)-(1.2). To do this, it will be useful to know when the operator  $\Delta$  has a continuous inverse. In the Neumann-type case (that is, when  $\alpha_0^\pm = 0$ ) it is clear that any constant function  $c$  lies in  $X$ , and  $\Delta c = 0$ , so  $\Delta$  cannot be invertible. Thus, to obtain invertibility it is necessary to exclude this case. In view of the assumption (1.4) (we still assume the basic hypotheses (1.4)-(1.6)), we can achieve this by imposing the further condition

$$\alpha_0^- + \alpha_0^+ > 0. \tag{4.1}$$

The following results are proved in [15, Theorem 2.1] and [15, Lemma 4.16].

**Theorem 4.1.** *Suppose that (4.1) holds. Then:*

- (a)  $\Delta : X \rightarrow Y$  has a bounded inverse  $\Delta^{-1} : Y \rightarrow X$ ;
- (b) *each eigenvalue  $\lambda_k$ ,  $k \geq 0$ , is a characteristic value of the operator  $-\Delta^{-1} : Y \rightarrow Y$ , with algebraic multiplicity 1.*

Theorem 4.1 will be required in the following sections, so from now on we will suppose that (4.1) holds (without explicitly restating this).

**Remark 4.2.** If  $\alpha_0^\pm = 0$  (that is, if (4.1) does not hold) then both the boundary conditions (1.2) are of Neumann-type. Such boundary conditions were considered in [14], and similar result to those below were obtained there. Hence, there is no significant loss of generality in assuming (4.1) here.

**4.1. Global bifurcation theory.** We now consider the bifurcation problem

$$-\Delta u = \lambda f(u), \quad (\lambda, u) \in \mathbb{R} \times X, \quad (4.2)$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, and we use the notation  $f : Y \rightarrow Y$  to denote the Nemitskii operator defined by  $f(u)(x) := f(u(x))$ ,  $x \in [-1, 1]$ , for  $u \in Y$ . We also suppose that  $f$  satisfies

$$\xi f(\xi) > 0, \quad \xi \in \mathbb{R} \setminus \{0\}, \quad (4.3)$$

and

$$0 < f_0 := \lim_{\xi \rightarrow 0} \frac{f(\xi)}{\xi} < \infty \quad (4.4)$$

(we assume that this limit exists). These assumptions imply that  $u \equiv 0$  is a solution of (4.2) for all  $\lambda \in \mathbb{R}$ ; such solutions will be called *trivial*. We will obtain some Rabinowitz-type global bifurcation results for the set of non-trivial solutions of (4.2).

Let  $\mathcal{S} \subset \mathbb{R} \times X$  denote the set of non-trivial solutions  $(\lambda, u)$  of (4.2), and let  $\overline{\mathcal{S}}$  denote the closure of  $\mathcal{S}$  in  $\mathbb{R} \times X$ . In the following results, for any  $k \geq 0$ , we will use the generic notation  $N_k$  to denote one of the nodal sets  $R_k$ ,  $S_k$  or  $T_k$ , and similarly for  $N_k^\nu$ ,  $\nu \in \{\pm\}$ .

**Lemma 4.3.** (a)  $\overline{\mathcal{S}} \cap (\mathbb{R} \times \{0\}) \subset \cup_{k=0}^\infty \{(\lambda_k/f_0, 0)\}$ .

(b) Suppose that, for some  $k \geq 0$ ,  $\psi_k \in N_k$ . Then there is a neighbourhood  $\mathcal{O}_k$  of  $(\lambda_k/f_0, 0)$  in  $\mathbb{R} \times X$  such that  $\mathcal{S} \cap \mathcal{O}_k \subset \mathbb{R} \times N_k$ .

*Proof.* Follow the proof of [10, Lemma 4.4]. □

For each  $k \geq 0$ , let  $\mathcal{C}_k$  denote the connected component of  $\overline{\mathcal{S}}$  containing the point  $(\lambda_k/f_0, 0)$ . We now have the following Rabinowitz-type global bifurcation result for the solution set of (4.2). Here, a *continuum* is a closed, connected set.

**Theorem 4.4.** For each  $k \geq 0$  the continuum  $\mathcal{C}_k \subset (0, \infty) \times X$ , and at least one of the following alternatives holds:

- (a)  $\mathcal{C}_k$  is unbounded in  $(0, \infty) \times Y$ ;
- (b)  $(\lambda_j/f_0, 0) \in \mathcal{C}_k$  for some  $j \geq 0$ ,  $j \neq k$ .

*Proof.* Combining part (b) of Theorem 4.1 with [8, Theorem 2.3] proves the result, with  $\mathcal{C}_k \subset \mathbb{R} \times C^1[-1, 1]$  in part (a). The continuity of the operator  $\Delta^{-1} \circ f : Y \rightarrow X$ , then shows that the continuum  $\mathcal{C}_k$  can be regarded as a continuum in  $\mathbb{R} \times X$ . To show that  $\mathcal{C}_k \subset (0, \infty) \times X$  we note that, by Theorem 4.1, the only solution  $(\lambda, u)$  of (4.2) with  $\lambda = 0$  is  $(\lambda, u) = (0, 0)$ , but by part (a) of Lemma 4.3,  $(0, 0) \notin \overline{\mathcal{S}}$ , which implies that  $\mathcal{C}_k \cap (\{0\} \times X) = \emptyset$ . Since  $\lambda_k > 0$ , it follows from connectedness that  $\mathcal{C}_k \subset (0, \infty) \times X$ . □

For the problem (4.2) with standard (single-point) boundary conditions, it is shown in [8] that, for each  $k \geq 0$ ,

$$\psi_k \in N_k \quad \text{and} \quad \mathcal{C}_k \setminus \{(\lambda_k/f_0, 0)\} \subset (0, \infty) \times N_k, \quad (4.5)$$

with  $N_k = S_k$ . That is, the nodal properties of the solutions are preserved along each continuum  $\mathcal{C}_k$ . Combining this with part (b) of Lemma 4.3 shows that alternative (b) in Theorem 4.4 cannot hold for this problem, so  $\mathcal{C}_k$  must be unbounded (see [8, Theorem 2.3]). Unfortunately, in the case of the multi-point boundary conditions (1.2), these properties may not hold in general, but if they do then we again obtain an unbounded continuum of solutions. In fact, we can obtain the following result.

**Theorem 4.5.** *Suppose that, for some  $k \geq 0$ , (4.5) holds for some nodal set  $N_k$ , and for every  $j \geq 0$ ,  $j \neq k$ , we have  $\psi_j \notin N_k$ . Then  $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$ , where*

$$\mathcal{C}_k^\pm = (\mathcal{C}_k \cap ((0, \infty) \times N_k^\pm)) \cup \{(\lambda_k/f_0, 0)\}, \quad (4.6)$$

*and each set  $\mathcal{C}_k^\pm$  is closed, connected and unbounded in  $(0, \infty) \times Y$ .*

*Proof.* The proof is similar to the combined proofs of [2, Theorem 4.5] and [2, Theorem 4.8], which considered the case of Dirichlet-type boundary conditions at one end-point.  $\square$

**Remark 4.6.** In the cases of the Dirichlet-type or Neumann-type boundary conditions considered in [2, 10, 12–14] it is shown that nodal properties of solutions of (4.2) are in fact always preserved along the bifurcating continua, that is, (4.5) holds for all  $k \geq 0$  (with  $N_k = S_k$  in the Neumann-type case and  $N_k = T_{k+1}$  in the Dirichlet-type case). Hence, for these boundary conditions Theorem 4.5 holds for all  $k$ , and we simply have the analogue of Rabinowitz' global bifurcation theorem [8, Theorem 2.3].

We have not shown here how one might verify that (4.5) holds. We will illustrate one approach to this in the following section, and use the results to obtain nodal solutions of the problem.

**4.2. Nodal solutions.** We now consider the problem

$$-\Delta u = f(u), \quad u \in X \quad (4.7)$$

(that is, the problem (1.1)-(1.2)), assuming that  $f$  satisfies the assumptions in Section 4.3, and also that the following limit exists

$$0 \leq f_\infty := \lim_{|\xi| \rightarrow \infty} \frac{f(\xi)}{\xi} \leq \infty \quad (4.8)$$

(it follows from (4.3) that  $0 \leq f_\infty$ ). We allow  $f_\infty = \infty$ , in which case we set  $1/f_\infty = 0$  below.

We are again interested in obtaining non-trivial solutions. In fact, we will use the global bifurcation results in Section 4.1 to obtain ‘nodal’ solutions of (4.7), that is, solutions  $u$  lying in specified nodal sets  $N_k^\nu$ . This will require preservation of nodal properties of solutions of (4.2) along the bifurcating continua, so we will need an analogue of Lemma 3.1 for solutions of (4.2). To obtain this we first note that if  $(\lambda, u)$  satisfies (4.2) then the following generalisation of (3.1) can be derived:

$$\lambda F(u(x)) + u'(x)^2 \equiv \lambda |F(u)|_0 = |u'|_0^2, \quad x \in [-1, 1], \quad (4.9)$$

where  $F : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$F(\xi) := 2 \int_0^\xi f(s) ds, \quad \xi \in \mathbb{R}.$$

Note that for the linear equation  $f(s) = s$ , so that  $F(\xi) = \xi^2$ , and (4.9) reduces to (3.1). Also, by (4.3), the function  $F$  is strictly increasing (respectively, decreasing) on  $[0, \infty)$  (respectively,  $(-\infty, 0]$ ). We now have the following generalisation of Lemma 3.1.

**Lemma 4.7.** *Suppose that  $(\lambda, u)$ ,  $\lambda > 0$ ,  $u \neq 0$ , satisfies (4.2).*

*(a) Suppose that, for some  $\gamma > 0$  and  $\nu \in \{\pm\}$ ,*

$$F(\xi) \leq \gamma \xi^2, \quad \xi \in \mathbb{R}, \quad (4.10)$$

$$\alpha_0^\nu > \sum_{i=1}^{m^\nu} |\alpha_i^\nu| + \lambda^{1/2} \gamma^{1/2} \sum_{i=1}^{m^\nu} |\beta_i^\nu|, \quad (4.11)$$

*then  $u'(\nu) \neq 0$ .*



(b) Suppose that, for some  $\gamma > 0$  and  $\nu \in \{\pm\}$ ,

$$F(\xi) \geq \gamma \xi^2, \quad \xi \in \mathbb{R}, \quad (4.12)$$

$$\beta_0^\nu > \frac{1}{\lambda^{1/2} \gamma^{1/2}} \sum_{i=1}^{m^\nu} |\alpha_i^\nu| + \sum_{i=1}^{m^\nu} |\beta_i^\nu|. \quad (4.13)$$

Then  $u(\nu) \neq 0$ .

*Proof.* The proof is similar to that of Lemma 3.1, using  $(1.2)^\nu$  and (4.9)-(4.13).  $\square$

We now obtain the desired nodal solutions of (4.7).

**Theorem 4.8.** Suppose that, for some  $k \geq 0$ ,

$$0 \leq f_\infty < \lambda_k < f_0. \quad (4.14)$$

- (a) Suppose that  $\psi_k \in T_{k+1}$ , and  $\psi_j \notin T_{k+1}$  for every  $j \neq k$  satisfying  $\lambda_j \leq f_0$ . Suppose also that (4.10) and (4.11) $^\pm$  hold with  $\lambda = 1$ , for some  $\gamma \geq f_0$ . Then (4.7) has solutions  $u_k^\pm \in T_{k+1}^\pm$ .
- (b) Suppose that  $\psi_k \in S_k$ , and  $\psi_j \notin S_k$  for every  $j \neq k$  satisfying  $\lambda_j \geq f_\infty$ . Suppose also that (4.12) and (4.13) $^\pm$  hold with  $\lambda = 1$ , for some  $0 < \gamma \leq f_\infty$ . Then (4.7) has solutions  $u_k^\pm \in S_k^\pm$ .

*Proof.* (a) We note that (4.14) is equivalent to

$$\lambda_k/f_0 < 1 < \lambda_k/f_\infty. \quad (4.15)$$

Now, by Theorem 4.4 there exists a continuum  $\mathcal{C}_k$  of solutions of (4.2) bifurcating from the point  $(\lambda_k/f_0, 0)$ . Also, by Lemma 4.3 and the results of [1] and [8],  $\mathcal{C}_k$  can be decomposed into two subcontinua  $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$ , each containing  $(\lambda_k/f_0, 0)$  and such that in a neighbourhood  $\mathcal{O}_k$  of  $(\lambda_k/f_0, 0)$ ,

$$(\mathcal{C}_k^\pm \setminus \{(\lambda_k/f_0, 0)\}) \cap \mathcal{O}_k \cap ((0, \infty) \times T_{k+1}^\pm) \neq \emptyset. \quad (4.16)$$

To find the desired solutions of (4.7) we will show that  $\mathcal{C}_k^\pm$  intersects the hyperplane  $\{1\} \times X$  at a pair of non-trivial solutions  $(1, u_k^\pm)$  of (4.2), with  $u_k^\pm \in T_{k+1}^\pm$ , and the functions  $u_k^\pm$  are then the desired solutions of (4.7). This type of argument is well-known for standard, single-point Sturm-Liouville boundary conditions, see, for example, the proof of [2, Theorem 5.3] for more details, although the argument predates [2]. The difficulty in the present situation is the potential non-preservation of the nodal properties of the solutions on the continua  $\mathcal{C}_k^\pm$ . That is, (4.6) may not hold globally, and the continua  $\mathcal{C}_k^\pm$  may not have the properties in Theorem 4.5.

To deal with this, suppose that for some  $\nu \in \{\pm\}$ ,  $\mathcal{C}_k^\nu \cap (\{1\} \times X) = \emptyset$ . Then, since  $\lambda_k/f_0 < 1$  and  $\mathcal{C}_k^\nu$  is connected, we have  $\mathcal{C}_k^\nu \subset (0, 1) \times X$ . We will show that

$$\mathcal{C}_k^\nu \setminus \{(\lambda_k/f_0, 0)\} \subset (0, 1) \times T_{k+1}^\nu. \quad (4.17)$$

We first note that since any non-trivial point  $(\lambda, u) \in \mathcal{C}_k^\nu$  has  $\lambda < 1$ , the hypotheses of part (a) of Lemma 4.7 hold at  $(\lambda, u)$ , which implies that  $u \notin \partial T_{k+1}^\nu$ . So, by (4.16) and the construction of the continua  $\mathcal{C}_k^\pm$  in [1], if (4.17) is false there must be a trivial point  $(\lambda_j/f_0, 0) \in \mathcal{C}_k^\nu$ , for some integer  $j \neq k$  with  $\lambda_j/f_0 \leq 1$  and  $\psi_j \in T_{k+1}^\nu$ . However, this contradicts the hypothesis in the theorem, so we conclude that (4.17) must be true. It now follows (similarly) from (4.16) and (4.17) that  $\mathcal{C}_k^+ \cap \mathcal{C}_k^- = (\lambda_k/f_0, 0)$ , so by [1, Theorem 2],  $\mathcal{C}_k^\nu$  is unbounded.

Standard arguments (see the proof of [2, Theorem 5.3]) now show that there exists a sequence of non-trivial points  $(\mu_n, v_n) \in \mathcal{C}_k^\nu$ ,  $n = 1, 2, \dots$ , such that, as  $n \rightarrow \infty$ ,

$$\begin{cases} \mu_n \rightarrow \lambda_k/f_\infty, & |v_n|_0 \rightarrow \infty, & \text{if } f_\infty > 0, \\ \mu_n \rightarrow \infty, & & \text{if } f_\infty = 0. \end{cases}$$

However, each of these alternatives contradicts (4.15) and (4.17), so  $\mathcal{C}_k^\nu \cap (\{1\} \times X) \neq \emptyset$ . Next, by similar arguments to those above, it can also be shown that there must be at least one non-trivial point in this intersection, which completes the proof of part (a).

(b) In this case we use bifurcation from infinity (see [9] for more details of this) to obtain a continuum  $\mathcal{D}_k = \mathcal{D}_k^+ \cup \mathcal{D}_k^-$  of solutions of (4.2) ‘bifurcating from  $(\lambda_k/f_\infty, \infty)$ ’, with similar properties to those of  $\mathcal{C}_k$  and  $\mathcal{C}_k^\pm$ . Now, in a similar manner to the proof of part (a), we can use part (b) of Lemma 4.7 to show that if  $\mathcal{D}_k^\nu$ ,  $\nu \in \{\pm\}$ , does not intersect the hyperplane  $\{1\} \times X$  then

$$(\lambda_k/f_0, 0) \in \mathcal{D}_k^\nu, \quad \mathcal{D}_k^\nu \setminus \{(\lambda_k/f_0, 0)\} \subset (1, \infty) \times S_k^\nu,$$

which again contradicts (4.15), and so yields solutions  $(1, u_k^\pm) \in \{1\} \times S_k^\pm$  of (4.2), and hence of (4.7).  $\square$

We can of course reverse the inequalities in (4.14).

**Theorem 4.9.** *Suppose that, for some  $k \geq 0$ ,*

$$f_0 < \lambda_k < f_\infty \leq \infty. \quad (4.18)$$

- (a) *Suppose that  $\psi_k \in T_{k+1}$ , and  $\psi_j \notin T_{k+1}$  for every  $j \neq k$  satisfying  $\lambda_j \leq f_\infty$ . Suppose also that (4.10) and (4.11) $^\pm$  hold with  $\lambda = 1$ , for some  $\gamma \geq f_\infty$ . Then (4.7) has solutions  $u_k^\pm \in T_{k+1}^\pm$ .*
- (b) *Suppose that  $\psi_k \in S_k$ , and  $\psi_j \notin S_k$  for every  $j \neq k$  satisfying  $\lambda_j \geq f_0$ . Suppose also that (4.12) and (4.13) $^\pm$  hold with  $\lambda = 1$ , for some  $\gamma \leq f_0$ . Then (4.7) has solutions  $u_k^\pm \in S_k^\pm$ .*

*Proof.* If  $f_\infty < \infty$  then the proof is similar to that of Theorem 4.8, so will not be repeated. If  $f_\infty = \infty$  then we follow the proof of Theorem 4.8, but modify it as in the proof of [2, Theorem 5.5] to obtain an unbounded sequence  $(\mu_n, u_n) \in \mathcal{C}_k^\nu$ ,  $n = 1, 2, \dots$ , such that  $\mu_n \rightarrow 0$ , from which the result follows as before (in this case,  $\mathcal{C}_k^\nu$  bifurcates from  $(\lambda_k/f_0, 0)$  with  $\lambda_k/f_0 > 1$ ).  $\square$

**Remark 4.10.** (a) The conditions (4.14), (4.18), say that the asymptotic gradients  $f_0$ ,  $f_\infty$  of  $f$  lie on either side of the eigenvalue  $\lambda_k$ , so the gradient of  $f$  ‘crosses’  $\lambda_k$ . This type of ‘crossing of eigenvalues’ condition is a standard condition used to obtain nodal solutions.

(b) In Theorems 4.8 and 4.9, the conditions on the values of  $\gamma$  in the inequalities (4.10) and (4.12) are related to the conditions (4.14) and (4.18) on the asymptotic values of  $f$ . Since

$$\lim_{\xi \rightarrow 0} \frac{F(\xi)}{\xi^2} = f_0, \quad \lim_{\xi \rightarrow \infty} \frac{F(\xi)}{\xi^2} = f_\infty,$$

we see that (4.10) can only hold with  $\gamma \geq \max\{f_0, f_\infty\}$ , while (4.12) can only hold with  $\gamma \leq \min\{f_0, f_\infty\}$ , so the hypotheses on  $\gamma$ ,  $f_0$  and  $f_\infty$  in these theorems are consistent.

(c) Some simple sufficient conditions for (4.10) and (4.12) to hold, with  $\gamma = f_0$ , are as follows: writing  $f$  in the form  $f(\xi) = (f_0 + g(\xi))\xi$ ,  $\xi \in \mathbb{R}$ , then

$$g \leq 0 \implies (4.10) \text{ holds,} \quad g \geq 0 \implies (4.12) \text{ holds.}$$

**Remark 4.11.** In the proofs of Theorems 4.8 and 4.9 we have obtained continua of solutions, but these do not have the full Rabinowitz-type global properties as described in Theorem 4.5, since the hypotheses imposed only ensure preservation of nodal properties above or below  $\lambda = 1$ , and the continua cross  $\lambda = 1$ , so the nodal properties need not be preserved globally. These hypotheses could be strengthened to yield such full global results in a variety of ways, but for brevity we will omit this here.

**4.3. A nonresonance condition.** Finally, we briefly consider the following ‘inhomogeneous’ form of (4.7),

$$-\Delta u = f(u) + h, \quad u \in X, \quad (4.19)$$

where  $f$  still satisfies the assumptions in Section 4.2, and  $h$  is an arbitrary function in  $Y$ .

**Theorem 4.12.** *Suppose that  $0 \leq f_\infty < \infty$  and  $f_\infty$  is not an eigenvalue of  $-\Delta$ . Then, for any  $h \in Y$ , equation (4.19) has a solution  $u \in X$ .*

*Proof.* The proof is similar to the proof of [11, Theorem 4.1], using the Leray-Schauder continuation theorem, in a relatively standard manner (given the properties of the operator  $\Delta^{-1}$  in Theorem 4.1).  $\square$

**Remark 4.13.** (a) Theorem 4.12 can be extended to a Sobolev space setting (instead of the above  $C^n$  setting), in a similar manner to that described in [2, Remark 5.2].

(b) The hypothesis in Theorem 4.12 that  $f_\infty$  is not an eigenvalue of  $-\Delta$  is a ‘nonresonance’ condition. Nonresonance conditions have been extensively investigated for a multitude of boundary value problems, including ordinary and partial differential equation problems.

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